Free probability techniques in quantum information theory

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Random quantum channels

- Quantum channels: completely positive, trace preserving maps

$$\Phi : \mathcal{M}_{\text{in}}(\mathbb{C}) \to \mathcal{M}_{\text{out}}(\mathbb{C}).$$

- Minimal Output Entropy of a quantum channel

$$H_{\min}(\Phi) = \min_{\rho \in \mathcal{M}_{\text{in}}(\mathbb{C})} H(\Phi(\rho)).$$

- Is the MOE additive ?

$$H_{\min}(\Phi \otimes \Psi) = H_{\min}(\Phi) + H_{\min}(\Psi) \quad \forall \Phi, \Psi.$$

- NO !!! Hayden '07, Hayden + Winter '08, Hastings '09.
- Counterexamples to additivity conjectures are random.
- Model of random quantum channels

$$\Phi(\rho) = \text{Tr}_{\text{anc}}(W \rho W^*),$$

where $W : \mathbb{C}^{\text{in}} \to \mathbb{C}^{\text{out}} \otimes \mathbb{C}^{\text{anc}}$ is a Haar random isometry.

How to get counterexamples ?

- Choose Φ to be random and $\Psi = \overline{\Phi}$ (conjugate channel, replace U by $\overline{U}$).
- Find lower bounds for $H_{\min}(\Phi) = H_{\min}(\overline{\Phi})$
  1. Hayden + Leung + Winter
  2. Hastings + Fukuda + King
- Find upper bounds for $H_{\min}(\Phi \otimes \overline{\Phi})$.

Strategy for $H_{\min}(\Phi \otimes \overline{\Phi})$

- Use trivial bound

$$H_{\min}(\Phi \otimes \overline{\Phi}) \leqslant H\left([\Phi \otimes \overline{\Phi}](X_{12})\right),$$

for a particular choice of $X_{12} \in \mathcal{M}_{\text{in}}(\mathbb{C}) \otimes \mathcal{M}_{\text{in}}(\mathbb{C})$.

- $X_{12} = X_1 \otimes X_2$ do not yield counterexamples $\Rightarrow$ choose a maximally entangled state $X_{12} = P_{\mathrm{Bell}} = E_{\text{in}}$.
- Use linear algebra to show that

$$\lambda_{\max}(Z) = \lambda_{\max}\left([\Phi \otimes \overline{\Phi}](E_{\text{in}})\right) \geqslant \frac{\text{in}}{\text{out} \cdot \text{anc}}.$$

Main result - finite rank output

Theorem (Collins + N. '09)

Let $\Phi : \mathcal{M}_{tnk} \to \mathcal{M}_k$ *be a random quantum channel (in* $= tnk$*, out* $= k$*, anc* $= n$*,* k *fixed,* $t \in (0,1)$ *fixed,* $n \to \infty$*). For all* k, t*, almost surely as* $n \to \infty$*, the eigenvalues of* $Z = [\Phi \otimes \overline{\Phi}](E_{tnk})$ *converge to*

$$\left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right).$$

- "Linear algebra" bound: for all t, n, k, the largest eigenvalue of Z is at least t.
- Two improvements:
  1. "better" largest eigenvalue,
  2. knowledge of the whole spectrum.
- Precise knowledge of eigenvalues $\rightsquigarrow$ optimal estimates for $H(Z)$.
- However, smaller eigenvalues are the "worst possible" (flat spectrum).
- Maximal violation for the p-Rényi MOE additivity conjecture for $t = 1/2$ (input coupled to a qubit).

Main result - unbounded rank

Theorem (Collins + N. '09)

Let $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ be a random quantum channel (in $= n$, out $= n$, anc $= k$, $n, k \to \infty$, $k/n \to c > 0$ fixed). For all $c > 0$, the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{n^2}$ of $Z = [\Phi \otimes \overline{\Phi}](E_n)$ satisfy:

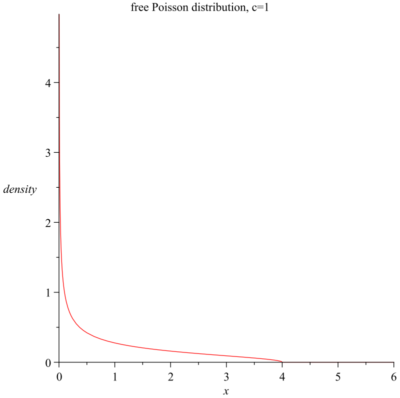
- *In probability, $cn\lambda_1 \to 1$.*
- *Almost surely, $\frac{1}{n^2-1} \sum_{i=2}^{n^2} \delta_{c^2 n^2 \lambda_i}$ converges to a free Poisson distribution of parameter c^2.*

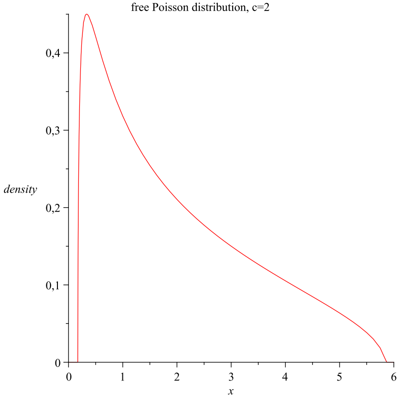
- Large eigenvalue $1/cn = 1/k$ due to $\Phi - \overline{\Phi}$ symmetry.
- New phenomenon in Random Matrix Theory: eigenvalues of two different magnitude orders (n^{-1} and n^{-2}).
- Smaller eigenvalues have non-trivial distribution.
- Precise knowledge of eigenvalues $\rightsquigarrow$ optimal estimate for $H(Z)$.
- Replace $\overline{\Phi}$ by Ψ, an independent random quantum channel from the same ensemble $\rightsquigarrow$ almost surely, $\frac{1}{n^2} \sum_{i=1}^{n^2} \delta_{c^2 n^2 \lambda_i}$ converges to a free Poisson distribution of parameter c^2 (all eigenvalues are of order n^{-2}).

Free Poisson distribution

Free Poisson (aka Marchenko-Pastur) distribution of parameter $c > 0$:

$$\pi_c = \max(1-c,0)\delta_0 + \frac{\sqrt{4c-(x-1-c)^2}}{2\pi x}\,\mathbf{1}_{[1+c-2\sqrt{c},1+c+2\sqrt{c}]}(x)\,dx.$$





Thank you !

http://arxiv.org/abs/0905.2313
http://arxiv.org/abs/0906.1877
http://arxiv.org/abs/0910.1768