

Positivity in Quantum Information Theory

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Entanglement in Quantum Information Theory

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- Non-separable states are called **entangled**.

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- Separability / entanglement are defined wrt a **fixed** partition $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. States separable wrt **any** partition are called **absolutely separable**

$$A\mathcal{SEP} = \bigcap_{U \in \mathcal{U}_d} U \cdot \mathcal{SEP} \cdot U^*.$$

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$$ASEP = \bigcap_{U \in \mathcal{U}_d} U \cdot SEP \cdot U^*.$$

- Separability of ρ_{12} depends on the eigenvalues and eigenvectors of ρ_{12} ; absolute separability depends only on the spectrum of ρ_{12} .

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Shor's quantum factoring algorithm

- Runs on a quantum computer with polynomial time $O(\log^3 N)$.
- Classical sieve algorithms run in sub-exponential time $O(\exp(\log^{1/3} N))$.
- Entanglement is necessary for the exponential speed-up.
- State of the art factorization : $15 = 3 \times 5$.

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- For rank one quantum states, entanglement can be detected and quantified by the von Neumann entropy

$$H(P_x) = S(\text{sv}(x)) = - \sum_i s_i(x) \log s_i(x), \quad x \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \cong \mathcal{M}_{d_1 \times d_2}(\mathbb{C}).$$

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- Detecting entanglement in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ is trivial via the PPT criterion [Horodecki].

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \rightarrow \mathcal{M}(\mathbb{C}^d)$ is called
 - **positive** if $A \geq 0 \implies f(A) \geq 0$;
 - **completely positive** if $f \otimes \text{id}_k$ is positive for all $k \geq 1$.

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- The transposition map t is an entanglement witness. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \mid t_{d_1} \otimes \text{id}_{d_2}(\rho_{12}) \geq 0\}.$$

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- For $(d_1, d_2) \in \{(2, 2), (2, 3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\text{Bell}}$, where

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- Written as a matrix in $\mathcal{M}^{1,+}(\mathbb{C}^4)$

$$\rho_{12} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

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- Partial transposition: transpose the block matrix $B = (B_{ij})$, keeping the blocks intact:

$$\rho_{12}^{\Gamma} = \mathfrak{t}_2 \otimes \text{id}_2(\rho_{12}) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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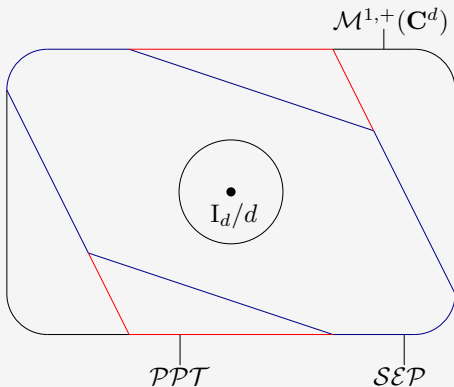
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- This matrix is no longer positive \implies the state is entangled.

A family of convex sets



- States in $\mathcal{PPT} \setminus \mathcal{SEP}$ are called **bound entangled**: no “maximal” entangled can be distilled from them.
- All these sets contain an open ball around the identity.

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]SE\mathcal{P}$, $[\mathcal{A}]PPT$, one needs a probability measure on the compact set $\mathcal{M}_d^{1,+}(\mathbb{C}^d)$ of positive, unit trace matrices.

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- Let $X \in \mathcal{M}_{d \times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\text{Tr}(XX^*)} = \frac{W_{d,s}}{\text{Tr}W_{d,s}}.$$

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- Almost surely, $\rho_{d,s}$ has full rank iff $s \geq d$.
- The measure $\mu_{d,s}$ is **unitarily invariant**: there exist a probability measure $\nu_{d,s}$ on the probability simplex $\Delta_d = \{\lambda \in \mathbb{R}^d \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$ such that if $\lambda \sim \nu_{d,s}$ and U is a Haar unitary matrix independent of λ ,

$$U \text{diag}(\lambda) U^* \sim \mu_{d,s}.$$

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Eigenvalues for induced measures

- Let X be a Gaussian matrix.

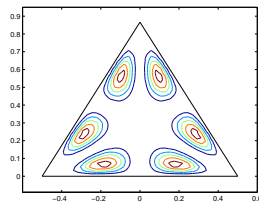
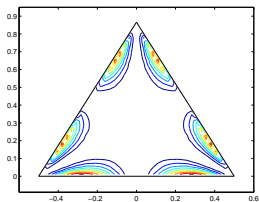


Figure: Induced measure eigenvalue distribution for $(d=3, s=3)$ and $(d=3, s=5)$.

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Volume of convex sets under the induced measure

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$$\lim_{s \rightarrow \infty} \mu_{d,s}(C) = 1.$$

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Definition

A pair of functions $(s_0(d), s_1(d))$ are called a **threshold** for a family of convex sets $\{C_d\}_{d \geq 2}$ if both conditions below hold

- If $s \lesssim s_0(d)$, then

$$\lim_{s \rightarrow \infty} \mu_{d,s}(C_d) = 0;$$

- If $s \gtrsim s_1(d)$, then

$$\lim_{s \rightarrow \infty} \mu_{d,s}(C_d) = 1.$$

Threshold values for $[A]SEP$, $[A]PPT$

Set	Balanced	Unbalanced
	$\mathbb{C}^d = \mathbb{C}^{\sqrt{d}} \otimes \mathbb{C}^{\sqrt{d}}$	$\mathbb{C}^d = \mathbb{C}^p \otimes \mathbb{C}^{d/p}$
SEP	$s_0 = Cd^{3/2}$ $s_1 = Cd^{3/2} \log^2 d$ [Aubrun, Szarek, Ye]	$s_0 = Cpd$ $s_1 = Cpd \log^2 d$ [Aubrun, Szarek, Ye]
PPT	$s_0 = s_1 = 4d$ [Aubrun]	$s_0 = \left[2 + 2\sqrt{1 - p^{-2}}\right] d$ Most likely $s_1 = s_0$ [Banica, N.]
$ASEP$?	?
	[Collins, N., Ye, in progress]	[Collins, N., Ye, in progress]
$APPT$	$s_0 = 64/(9\pi^2)d^2$ $s_1 = 4d^2$ [Collins, N., Ye]	$s_0 = s_1 = \left[p + \sqrt{p^2 - 1}\right]^2 d$ [Collins, N., Ye]

PPT, unbalanced case

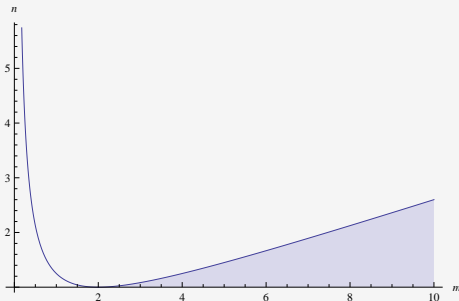
Theorem (Banica, N. - arXiv:1105.2556)

Let W be a complex Wishart matrix of parameters (dn, dm) . Then, with $d \rightarrow \infty$, the empirical spectral distribution of mW^Γ converges in moments to a free difference of free Poisson distributions of respective parameters $m(n \pm 1)/2$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$n \leq \frac{m}{4} + \frac{1}{m} \quad \text{and} \quad m \geq 2.$$



PPT, unbalanced case

What's a free difference of free Poisson distributions ?

- The free Poisson distribution of parameter $c > 0$:

$$\pi_c = \max(1 - c, 0)\delta_0 + \frac{\sqrt{4c - (x - 1 - c)^2}}{2\pi x} \mathbf{1}_{[1+c-2\sqrt{c}, 1+c+2\sqrt{c}]}(x) dx.$$

- Free additive convolution of two compactly supported probability distributions $\mu_{1,2}$: sample $X_{1,2} \in \mathbb{R}^n$ from $\mu_{1,2}$ and consider

$$A = U_1 \text{diag}(X_1) U_1^* + U_2 \text{diag}(X_2) U_2^*,$$

where $U_{1,2}$ are $n \times n$ independent Haar unitary matrices. Then, almost surely when $n \rightarrow \infty$, the spectrum of A has distribution $\mu_1 \boxplus \mu_2$.

Theorem

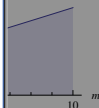
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APPT, unbalanced case

Theorem (Collins, N., Ye - soon on arXiv)

Let ρ be a random quantum state from the induced ensemble of parameters (d, s) . Almost surely, when $d \rightarrow \infty$ and $s \sim cd$, one has:

- If $c > (p + \sqrt{p^2 - 1})^2$, then ρ is APPT.
- Conversely, if $c < (p + \sqrt{p^2 - 1})^2$, then ρ is not APPT.

Theorem (Collins, N., Ye - soon on arXiv)

Let ρ be a random quantum state from the induced ensemble of parameters (d, s) . Then, for all $\varepsilon > 0$, almost surely, when $d \rightarrow \infty$ and $s > (4 + \varepsilon)p^2d$, the quantum state ρ is APPT. The following converse holds:

- When $1 \ll p^2 \ll d$ and $s < (4 - \varepsilon)p^2d$, ρ is not APPT.
- When $p^2 \sim \tau d$ for a constant $\tau \in (0, 1]$, there exists an explicitly computable constant C_τ such that whenever $s < 4(C_\tau - \varepsilon)p^2d$, ρ is not APPT.

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- The “absolute” (basis independent) formulation of the problem are considered.

Merci !