Random quantum channels
- graphical calculus -

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Random quantum channels
&
additivity problems

Additivity for MOE of quantum channels

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- NO !!!
  - $p > 1$: Hayden '07, Hayden + Winter '08
  - $p = 1$: Hastings '09

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- Equivalently, via the Stinespring dilation theorem

$$\Phi(\rho) = \mathrm{Tr}_{\mathrm{aux}}(U(\rho \otimes P_y)U^*),$$

where $y \in \mathbb{C}^{\frac{\mathrm{out} \times \mathrm{aux}}{\mathrm{in}}}$ and $U \in \mathcal{M}_{\mathrm{out} \times \mathrm{aux}}(\mathbb{C})$ is a Haar unitary matrix.

Our model

Choice of parameters

- in $= tnk$,
- out $= k$,
- aux $= n$,

where $n, k \in \mathbb{N}$ and $t \in (0,1)$. In general, we shall assume that

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We are thus considering random channels

$$\begin{aligned} \Phi : \mathcal{M}_{tnk}(\mathbb{C}) &\to \mathcal{M}_k(\mathbb{C}) \\ \rho &\mapsto \operatorname{Tr}_n \left[U(\rho \otimes P_y) U^* \right], \end{aligned}$$

where $y \in \mathbb{C}^{t^{-1}}$ is fixed (and irrelevant) and $U \in \mathcal{U}(nk)$ is a Haar random unitary matrix.

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Strategy

- Use trivial bound

$$H^p_{\min}(\Phi \otimes \overline{\Phi}) \leqslant H^p\left([\Phi \otimes \overline{\Phi}](X_{12})\right),$$

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- $X_{12} = X_1 \otimes X_2$ do not yield counterexamples $\Rightarrow$ choose a maximally entangled state $X_{12} = E_{tnk}$.
- Bound entropies of the (random) density matrix

$$Z = [\Phi \otimes \overline{\Phi}](E_{tnk}) \in \mathcal{M}_{k^2}(\mathbb{C}).$$

Main result for product channel

Theorem (Collins + N. '09)

For all k, t, almost surely as $n \to \infty$, the eigenvalues of $Z = [\Phi \otimes \overline{\Phi}](E_{tnk})$ converge to

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  1. "better" largest eigenvalue,
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- However, smaller eigenvalues are the "worst possible".
- Precise knowledge of eigenvalue $\rightsquigarrow$ optimal estimates for entropies.

Graphical calculus for random quantum channels

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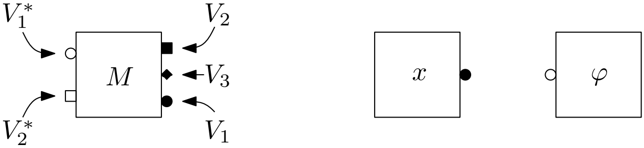

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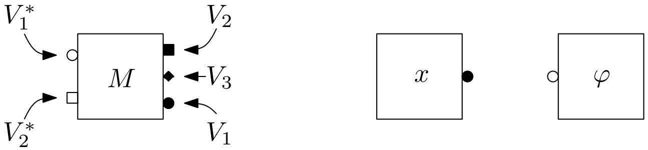

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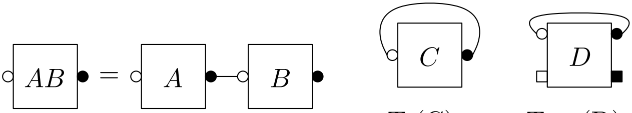

$\mathrm{Tr}(\mathrm{C})$ $\qquad \mathrm{Tr}_{\mathrm{V}_1}(\mathrm{D})$

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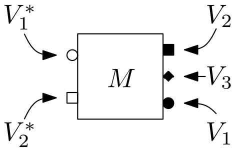

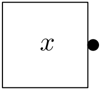

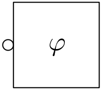

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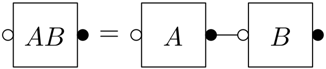
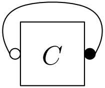
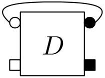


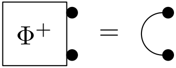


$\mathrm{Tr(C)}$



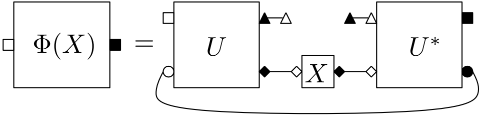
$\mathrm{Tr_{V_1}(D)}$

- Bell state $\Phi^+ = \sum_{i=1}^{\dim V_1} e_i \otimes e_i \in V_1 \otimes V_1$

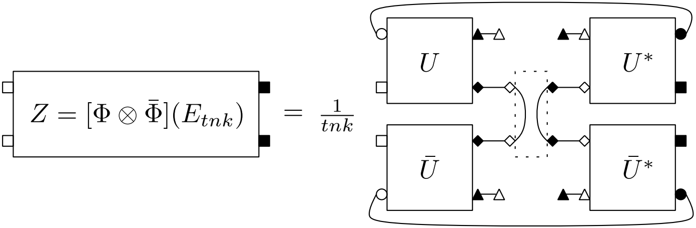


Graphical representation of quantum channels

- Single channel



- Product of conjugate channels



- Decorations/labels

●/○ $= \mathbf{C}^n$ ■/□ $= \mathbf{C}^k$ ◆/◇ $= \mathbf{C}^{tnk}$ ▲/△ $= \mathbf{C}^{t^{-1}}$

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- Example

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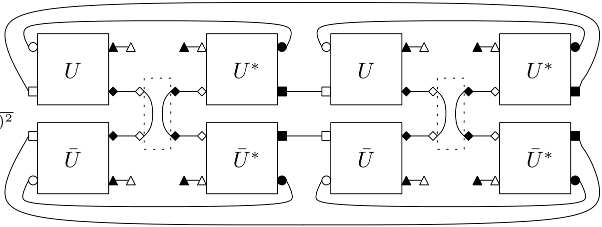

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Theorem (Weingarten formula)

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$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}} \, dU = \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(d, \alpha\beta^{-1}).$$

If $p \neq p'$ then

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- There is a graphical way of reading this formula on the diagrams !

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4. For all $i = 1, \ldots, p$, add a wire between each white decoration of the i-th U box and the corresponding white decoration of the $\alpha(i)$-th $\overline{U}$ box. In a similar manner, use β to pair black decorations.

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5. Erase all U and $\overline{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

Theorem

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta} \mathcal{D}_{(\alpha,\beta)} \,\mathrm{Wg}(d, \alpha\beta^{-1}).$$

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- After doing the loop combinatorics, one is left with maximizing over S_{2p}^2 quantities such as

$$\#(\gamma^{-1}\alpha) + \#(\alpha^{-1}\beta) + \#(\beta^{-1}\delta),$$

  where γ and δ are permutations coding the initial wiring of $U/\overline{U}$ boxes and $\#(\cdot)$ is the number of cycles function.

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$$\#(\gamma^{-1}\alpha) + \#(\alpha^{-1}\beta) + \#(\beta^{-1}\delta),$$

  where γ and δ are permutations coding the initial wiring of $U/\overline{U}$ boxes and $\#(\cdot)$ is the number of cycles function.
- Geodesic problems in symmetric groups $\Rightarrow$ non-crossing partitions $\Rightarrow$ free probability.

Application: product conjugate channel

- We want to compute, for all $p \geqslant 1$, $\mathbb{E}\,\mathrm{Tr}(Z^p)$.
- One needs to compute the contribution of each diagram $\mathcal{D}_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathcal{S}_{2p}$.
- $\mathcal{D}_{(\alpha,\beta)}$ is a collection of loops associated to vector spaces of dimensions n, k and tnk.
- After doing the loop combinatorics, one is left with maximizing over S_{2p}^2 quantities such as

$$\#(\gamma^{-1}\alpha) + \#(\alpha^{-1}\beta) + \#(\beta^{-1}\delta),$$

  where γ and δ are permutations coding the initial wiring of $U/\overline{U}$ boxes and $\#(\cdot)$ is the number of cycles function.
- Geodesic problems in symmetric groups $\Rightarrow$ non-crossing partitions $\Rightarrow$ free probability.
- Asymptotic for Weingarten weights:

$$\mathrm{Wg}(d,\sigma) = d^{-(p+|\sigma|)}(\mathrm{Mob}(\sigma) + O(d^{-2})).$$

Concluding remarks

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- Other applications to QIT (work in progress with B. Collins and K. Życzkowski)

Thank you !

Next talk $\rightsquigarrow$ bounds for 1 channel

`http://arxiv.org/abs/0905.2313`
`http://arxiv.org/abs/0906.1877`